NEW INVARIANTS IN TOPOLOGY
homological spectral package.

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Basic topological invariants

Fix a field κ

Space X $\quad\therefore$ (1.) $\boxed{H_r(X), \quad \beta_r(X) = \dim H_r(X)}$

Pair $(X; \xi \in H^1(X; \mathbb{Z}))$ $\quad\therefore$ (2.) $\boxed{H_r^N(X, \xi), \ \beta_r^N(X, \xi)}$

$\therefore$ (3.) Alexander Polynomial(s)

$\boxed{A_r(X, \xi)(z)}$ ($r-$ monodromy)

Purpose of the lecture

Provide REFINMENTS (for (1.) and (2.) and (3.)) called

HOMOLOGY SPECTRAL PACKAGE and **JORDAN BLOCKS**

in the presence of a continuous map $f : X \to \mathbb{R}$ or $f : X \to \mathbb{S}^1$ in analogy with

the SPECTRAL PACKAGE of $(V, T : V \to V)$ and
the JORDAN DECOMPOSITION of T in finite dimensional linear algebra.

SYSTHEM $(V, T : V \to V)$

$$\begin{cases} V \text{ f.d. complex vector space.} \\ T : V \to V \text{ linear map} \end{cases} \Longrightarrow$$

SPECTRAL PACKAGE

$$\begin{cases} \dim V = n & \in \mathbb{N} \\ z_1, z_2, \cdots, z_{k-1}, z_k & \in \mathbb{C}\text{;eigenvalues} \\ n_1, n_2, \cdots, n_{k-1}, n_k & \subseteq \mathbb{N}\text{;multiplicities} \\ V_1, V_2, \cdots, V_{k-1}, V_k & \subseteq V\text{;generalized eigenspaces} \end{cases}$$

PROPERTIES $\dim V = \sum n_i,\ \dim V_i = n_i,\ \ V = \oplus V_i$

A Jordan block:

$$T(\lambda,k)=\begin{pmatrix}\lambda & 1 & 0 & \cdots & 0\\ 0 & \lambda & 1 & \ddots & \vdots\\ 0 & 0 & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \lambda & 1\\ 0 & \cdots & 0 & 0 & \lambda\end{pmatrix}. \tag{1}$$

JORDAN DECOMPOSITION . With respect to some base in V any $T:V\to V$ is

$$\begin{pmatrix}T(\lambda_1,k_1) & 0 & 0 & 0\\ 0 & T(\lambda_2,k_2) & 0 & \vdots\\ 0 & 0 & \ddots & 0\\ 0 & \cdots & 0 & T(\lambda_r,k_r)\end{pmatrix}. \tag{2}$$

$$\boxed{\delta^T :\equiv \begin{cases} z_1, z_2, \cdots, z_{k-1}, z_k \\ n_1, n_2, \cdots, n_{k-1}, n_k \end{cases}} \Rightarrow \begin{cases} \text{finite configurations of} \\ \text{points with multiplicities} \\ \text{s.t.} \sum_i \delta^T(z_i) = \dim V \end{cases}$$

$\delta^T \equiv P^T(z) = (z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_k)^{n_k}$
the *characteristic polynomial*
a degree $n-$ monic polynomial, $n = \dim V$.

$$\boxed{\hat{\delta}^T :\equiv \begin{cases} z_1, z_2, \cdots, z_{k-1}, z_k \\ V_1, V_2, \cdots, V_{k-1}, V_k \end{cases}} \Rightarrow \begin{cases} \text{finite configurations of} \\ \text{disjoint subspaces of V} \\ \text{s.t. } \oplus_i \hat{\delta}^T(z_i) = V \end{cases}$$

basic properties

1. (Stability) $L(V, V) \ni T \rightsquigarrow \delta^T(z) = P^T(z) \in \mathbb{C}^n$ is continuous
2. (Duality) $\delta^T = \delta^{T^*}$
3. For an open and dense set of $T \in L(V, V)$, $\delta^T(z) = 0 \text{ or } 1$
4. (Computability) $P^T(z)$ can be calculated with arbitrary accuracy.

One regards $\delta^T \equiv P^T(z)$ as a *refinement* of dim V,

One regards $\hat{\delta}^T$ as an *implementation* of the refinement δ^T.

SYSTHEM

$(X, f : X \to \mathbb{R})$

$$\begin{cases} X \text{ a compact ANR}, \\ f \text{ continuous}, \\ \kappa \text{ a field}, \mathrm{H_r(X) := H_r(X;\kappa), r = 0, 1, \cdots} \dim \mathrm{X}. \end{cases}$$

HOMOLOGICAL SPECTRAL PACKAGE

$$\begin{cases} \dim H_r(X) = \beta_r(X)\ ; \quad \text{Betti number} \\ z_1, z_2, \cdots, z_{k-1}, z_k \in \mathbb{C}; \text{barcodes} \\ n_1, n_2, \cdots, n_{k-1}, n_k \in \mathbb{N}; \text{multiplicities} \\ V_1, V_2, \cdots, V_{k-1}, V_k; \text{quotients of subspaces of } \mathrm{H_r(X)}. \end{cases}$$

$\beta_r = \sum_i n_i,\ \dim V_i = n_i, H_r(X) \simeq \oplus_i V_i$

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$$\beta_r = \sum_i n_i,\ \dim V_i = n_i, H_r(X) \simeq \oplus_i V_i$$

specifications

- $V_i = L_i / L'_i$ quotients of subspaces of $H_r(X)$, $L'_i \subset L_i \subseteq H_r(X)$, s.t. $L_i \cap L_j = L'_i \cap L'_j$
- If $H_r(X)$ has a inner product then V_i is *canonically* a subspace of $H_r(X)$ with $V_i \perp V_j$ when $i \neq j$.

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$\delta^f_r \equiv P^f_r(z) = (z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_k)^{n_k}$
the *homological characteristic polynomial*
a degree $\beta_r(X)-$ monic polynomial

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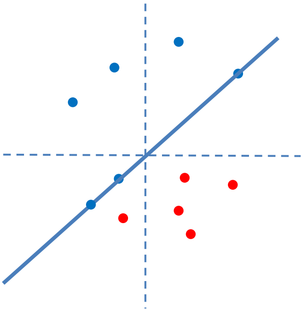

Notations

For $f : X \to \mathbb{R}$ and $a,\ b,\ a \leq b$ denote by :

- $X(a) := f^{-1}(a), \quad X_{[a,b]} := f^{-1}([a,b])$
- $Cr(f) = \{t \in \mathbb{R} \mid \text{homology of X(t) changes}\}$
- $supp\ \ \delta^f_r = \{z = a + ib \in \mathbb{C} \mid \delta^f_r(a,b) \neq 0\}$

$$H_r(X(a)) \xrightarrow{i_a^{[a,b]}} H_r(X_{[a,b]}) \xleftarrow{i_b^{[a,b]}} H_r(X(b))$$

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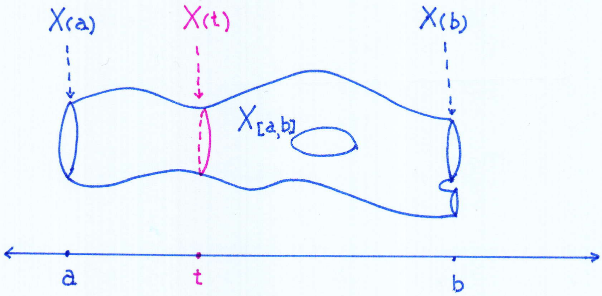

X(a)
X(t)
X(b)
X[a,b]
a
t
b

Meaning 2

Definition

- $x \in H_r(X(a))$ is *observable* at b if $i_a^{[a,b]}(x) \in \mathrm{i}mg\ i_b^{[a,b]}$

  $y \in H_r(X(b))$ is *observable* at a if $i_b^{[a,b]}(y) \in \mathrm{i}mg\ i_a^{[a,b]}$
- $x \in H_r(X(a))$ is *dead* at b if $i_a^{[a,b]}(x) = 0$

  $y \in H_r(X(b))$ is *dead* at a if $i_b^{[a,b]}(x) = 0$

- $z \in supp\ \delta^f_r$. If $z = a + ib, a \leq b$ then :
  - $a, b \in Cr(f)$ and for any $t \in [a, b]$
  - there exists $x \in H_r(X(t))$, observable at b and not at any $t'' > b$ and observable at a and not at any $t' < a$.

$\delta^f_r(z) = k$ means:

for any $t \in [a, b]$ there are exactly k linearly independent such x's.

- $z \in supp\ \delta^f_r$. If $z = a + ib, a \leq b$ then:
  - $a, b \in Cr(f)$ and for any $t \in (b, a)$
  - there exists $x \in H_{r-1}(X(t))$ dead at b and not at $t' < a$ and dead at a and not at $t' > a$.

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MEANINGS 3

- $z \in \mathrm{s}upp\ \delta^f_r$. If $z = a + ib, a \leq b$ then :
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  $\delta^f_r(z) = k$ means

  for any $t \in [a, b]$ there are exactly k linearly independent such x's.

Main results

Theorem

1. $\sum_{z \in \mathrm{supp} \delta^f_r} \delta^f_r(z) = \beta_r(X)$
2. *For an open sense set of maps* f, $\delta^f_r(z) = 0$ *or* $\delta^f_r(z) = 1$.

Theorem

The assignment $C(X, \mathbb{R}) \ni f \rightsquigarrow \delta^f_r \in \mathbb{C}^{\beta_r}$ *is continuous*

(The same remains true for the configuration $\hat{\delta}^f_r$)

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Theorem

If M^n is a closed $\kappa-$ orientable topological $n-$dimensional manifold then

$$\delta^f_r(a+ib) = \delta^{-f}_{n-r}(-b-ia)$$

(The same remains true for the configuration $\hat\delta^f_r$)

description of δ^f_r and $\hat{\delta}^f_r$

For $f : X \to \mathbb{R}$, and $a, b \in \mathbb{R}$

Denote:

$X_a := f^{-1}((-\infty, a])$ sub-level

$X^b := f^{-1}([b, \infty))$ over-level

Define:

- $\mathbb{I}_a(r) := \mathrm{i}mg(H_r(X_a) \to H_r(X))$
- $\mathbb{I}^b(r) := \mathrm{i}mg(H_r(X^b) \to H_r(X))$
- $\mathbb{F}_r(a, b) := \mathbb{I}_a \cap \mathbb{I}^b$

Then $a \leq a', b' \leq b$ imply $\mathbb{F}_r(a', b') \subseteq \mathbb{F}_r(a, b)$.

δ^f_r and $\hat{\delta}^f_r$; (2)

For $\epsilon > 0$ define

$$\boxed{\mathbb{F}_r(a, b, \epsilon) := \mathbb{F}_r(a, b)/\mathbb{F}_r(a - \epsilon, b) + \mathbb{F}_r(a, b + \epsilon)}$$

and observe that $\epsilon' > \epsilon''$ induces the surjective map

$$\boxed{\mathbb{F}_r(a, b; \epsilon') \to \mathbb{F}_r(a, b; \epsilon'').}$$

Definition

$$\hat{\delta}^f_r(a, b) := \varinjlim_{\epsilon \to 0} \mathbb{F}(a, b, \epsilon)$$

$$\delta^f_r(a, b) := \dim \hat{\delta}^f_r(a, b)$$

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Technical results 1

Box

$$\boxed{B = (a', a] \times [b, b')}$$

$$-\infty \le a' \le a \le \infty \; ; \; -\infty \le b \le b' \le \infty$$

Define

$$\boxed{\mathbb{F}_r(B) = \mathbb{F}_r(a,b)/\mathbb{F}_r(a',b) + \mathbb{F}_r(a,b')}$$

Note that for $a < \inf f$, $b > \sup f$

1. $\mathbb{F}_r(-\infty, b) = \mathbb{I}^b(r)$,
2. $\mathbb{F}_r(a, \infty) = \mathbb{I}_a(r)$
3. $\mathbb{F}_r(-\infty, a] \times [b, \infty)) = \mathbb{F}_r(a, b)$
4. $\mathbb{F}_r(-\infty, \infty) = \mathbb{F}_r(a, b)$

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4. $\mathbb{F}_r(-\infty, \infty) = \mathbb{F}_r(a, b)$

Proposition

Suppose X a compact ANR. Then

1. $\delta^f_r(a,b) \neq 0 \;\Rightarrow a, b$ *(homological) critical values;*
2. *For any B as above*

$$\mathbb{F}_r(B) \equiv \bigoplus_{(a,b)\in supp\ \delta^f_r \cap B} \hat{\delta}^f_r(a,b)$$

In particular $\beta_r(X) = \sum_{(a,b)\in supp\ \delta^f_r} \delta^f_r(a,b)$, *and*

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In particular $\beta_r(X) = \sum_{(a,b)\in \mathrm{s}upp\ \delta^f_r} \delta^f_r(a,b)$, *and*

Denote by $D(a, b; \epsilon) := (a - \epsilon, a + \epsilon) \times (b - \epsilon, \pm\epsilon)$

Proposition

Suppose X compact and $\epsilon < \epsilon(f)/3$. For any $g : X \to \mathbb{R}$ continuous map with $||f - g||_\infty < \epsilon$ and $a, b \in Cr(f)$ one has

$$\sum_{z \in D(a,b;2\epsilon)} \delta^g_r(z) = \delta^f_r(a, b)$$

$$\mathrm{supp}\,\delta^g_r \subset \bigcup_{z \in \mathrm{supp}\,\delta^f_r} D(a, b; 2\epsilon)$$

Real valued versus anglevalued maps

$f : X \to \mathbb{R}$

$\beta_r(X)$

$\delta^f_r =$ configuration of points in $\mathbb{R}^2 = \mathbb{C}$

$H_r(X)$

$f : X \to \mathbb{S}^1$

$\xi_f \in H^1(X : \mathbb{Z})$

$\beta^N_r(X; \xi_f)$

$\delta^f_r =$ configuration of points in $\mathbb{C} \setminus 0$

$H^N_r(X; \xi_f)$

$\kappa = \mathbb{C}$

$H_r(X; \mathbb{C})$

$\hat{\delta}^f_r =$ configuration of subspaces in $H_r(X; \mathbb{C})$

$H^{L_2}_r(\tilde{X})$ - HM over $L^\infty(S^1)$

$\hat{\delta}^f_r =$ configuration of of sub HM in $H^{L_2}_r(\tilde{X})$

Applications

In view of effective computability of the configuration δ^f_r the result can be used to :

- **Applications in topology:** Calculation of Betti numbers, Refinements of Morse inequalities .
- **Applications in data analysis:** Homological recognition of shapes which can be manifolds. Homological differentiations of shapes.
- **Applications in geometric analysis:** Refinement of Hodge de Rham theorem on compact Riemannian maniflods, Canonical base in the space of Harmonic forms
- **Applications in dynamics:** to dynamics which admit a Lyapunov functions.

Considerably more when combined with similar refinements of angle valued maps.

SYSTHEM

$(X, f : X \to \mathbb{S}^1) \to \xi_f \in H^1(X;\mathbb{Z})$

$$\begin{cases} X \text{ a compact ANR,} \\ f \text{ continuous,} \tilde{f} \text{ infinite cyclic cover} \\ \kappa \text{ a field, } H^N_r(X;\xi_f) := H_r(\tilde{X})/TH_r(\tilde{X}) \\ r = 0, 1, \cdots \dim X. \end{cases}$$

- $\tilde{X}$ the infinite cyclic cover associated to ξ_f
- $H_r(\tilde{X})$ a $\kappa[t^{-1}, t]-$ module
- $\kappa[t^{-1}, t]$ is PID and integral domain
- $TH_r(\tilde{X})$ is the torsion submodule of $H_r(\tilde{X})$
- $H^N_r(X;\xi) := H_r(\tilde{X})/TH_r(\tilde{X})$ a free $\kappa[t^{-1}, t]$module
- $\beta^N_r(X) := \mathrm{rank}\ \mathrm{H}^N_r(X;\xi)$

HOMOLOGICAL SPECTRAL PACKAGE

$$\begin{cases} \mathrm{rank}\ \mathrm{H}^{\mathrm{N}}_{\mathrm{r}}(\mathrm{X};\xi_{\mathrm{f}}) = \beta^{\mathrm{N}}_{\mathrm{r}}(\mathrm{X};\xi_{\mathrm{f}})\ ;\ \ \mathrm{Novikov-Betti\ number} \\ z_1, z_2, \cdots, z_{k-1}, z_k \in \mathbb{C} \setminus 0;\ \mathrm{barcodes} \\ n_1, n_2, \cdots, n_{k-1}, n_k \in \mathbb{N};\ \mathrm{multiplicities} \\ V_1, V_2, \cdots, V_{k-1}, V_k;\ \mathrm{free}\ \kappa[\mathrm{t}^{-1}, \mathrm{t}] - \mathrm{modules.} \end{cases}$$

$V_i = L_i/L'_i$ quotients of free $\kappa[t^{-1}, t]-$submodules of $H^N_r(X;\xi_f)$, $L'_i \subset L_i \subseteq H_r(X)$, s.t. $L_i \cap L_j = L'_i \cap L'_j$

$$\beta_r^N = \sum_i n_i, \ \mathrm{rank}\ V_i = n_i, H_r^N(X;\xi_f) \simeq \oplus V_i$$

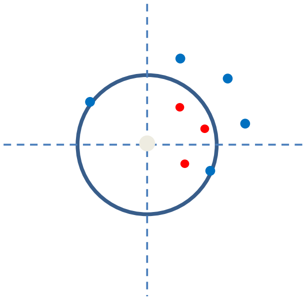
$$\delta_r^f(e^{(b-a)+ia}) := \delta^{\tilde f}(a,b)$$

$$\hat\delta_r^f(e^{(b-a)+ia}) := \bigoplus_{n\in\mathbb{Z}} \hat\delta^{\tilde f}(a+2\pi n, b+2\pi n)$$

If $\kappa = \mathbb{C}$ and a $\kappa[t^{-1}, t]-$inner product

By von Neumann completion

- $H_r^N(X;\xi) \rightsquigarrow H_r^{L_2}(\tilde{X})$ an $L^\infty(S^1)-$Hilbert module
- $V_i \rightsquigarrow$ orthogonal closed $L^\infty(\mathbb{S}^1)-$ Hilbert submodules
- $\beta_r^N(M;\xi_f) = \dim_{vN} H_r^{L_2}(\tilde{X})$



An additional invariant:
The Monodromy:

The $\kappa[t^{-1}, t]-$module $\mathbb{V}_r := TH_r(\tilde{X})$

1. $\mathbb{V}_r$ is f.d. $\kappa-$vector space.
2. $T : \mathbb{V}_r \to \mathbb{V}_r$ is a linear isomorphism

Described and discussed in the next talk.